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Some Norms of Toeplitz Matrices with the Generalized Balancing Numbers

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Abstract. In this article, we present results on Toeplitz matrices whose input elements are balancing numbers. First, the Toeplitz matrices whose elements are the balancing numbers are created and then the Euclidian, row and column norms of these matrices are found. Furthermore lower and upper bounds are obtained for the spectral norms of these matrices. In addition, the upper bounds for the Frobenius (Euclidian) and spectral norms of the Kronecker and Hadamard product matrices of the Toeplitz matrices with the balancing numbers are calculated.

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1. Introduction

There have been several articles published on the norms of special matrices, specifically Toeplitz matrices, with different number sequences. In these articles, authors have investigated the spectral norms of these matrices, as well as provided lower and upper bounds for their spectral norms.

We now give a short literature review on Toeplitz matrices with numbers having linear recurrence relations. Solak [5] focused on calculating the spectral norms of Toeplitz matrices with Fibonacci and Lucas numbers. Akbulak and Bozkurt [1] obtained special norms of Toeplitz matrices given with Fibonacci and Lucas numbers and derived lower and upper bounds for the spectral norm. Shen [4] obtained special norms for Toeplitz matrices with k-Fibonacci and k-Lucas numbers. They also provided bounds for the spectral norms of these matrices and lower and upper bounds for the spectral norms of Hadamard and Kronecker products involving these matrices.

Eylem G. Karpuz [3] studied the norms of Toeplitz matrices with elements represented by Pell numbers. Daşdemir [2] explored special norms of Toeplitz matrices, such as Pell, Pell-Lucas, and modified Pell numbers. He derived lower and upper bounds for the spectral norm. Uygun [8] focused on Toeplitz matrices with

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Jacobsthal and Jacobsthal-Lucas numbers. She obtained special norms, lower and upper bounds for the spectral norm, and the upper bound of the Frobenius norm for the Kronecker and Hadamard products of these matrices. Uygun [9] also conducted a parallel study on the k-jacobsthal and k-jacobsthal-lucas numbers.

These researchers have contributed to the understanding of the spectral norms and bounds of Toeplitz matrices with various number sequences. In this paper, we obtain some special norms of Toeplitz matrices with balancing numbers. First, we present information on generalized Balancing sequence and its special cases.

A generalized balancing sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1)\}_{n\geq 0}$ is defined by the second-order recurrence relation

$$W_n = 6W_{n-1} - W_{n-2} \tag{1.1}$$

with the initial values $W_0 = c_0, W_1 = c_1$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 6W_{-(n-1)} - W_{-(n-2)}$$

for $n = 1, 2, 3, \cdots$. Therefore, recurrence equation (1.1) holds for all integer n.

The first few generalized balancing numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized balancing numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$6W_0 - W_1$
2	$6W_1 - W_0$	$35W_0 - 6W_1$
3	$35W_1 - 6W_0$	$204W_0 - 35W_1$
4	$204W_1 - 35W_0$	$1189W_0 - 204W_1$
5	$1189W_1 - 204W_0$	$6930W_0 - 1189W_1$
6	$6930W_1 - 1189W_0$	$40391W_0 - 6930W_1$
7	$40391W_1 - 6930W_0$	$235416W_0 - 40391W_1$
8	$235416W_1 - 40391W_0$	$1372105W_0 - 235416W_1$
9	$1372105W_1 - 235416W_0$	$7997214W_0 - 1372105W_1$
10	$7997214W_1 - 1372105W_0$	$46611179W_0 - 7997214W_1$

For more information on generalized balancing numbers, see for example, Soykan [6].

Balancing sequence $\{B_n\}_{n\geq 0}$, modified Lucas-balancing sequence $\{H_n\}_{n\geq 0}$ and Lucas-balancing sequence $\{C_n\}_{n\geq 0}$ are defined respectively, by the second order recurrence relations;

$$B_n = 6B_{n-1} - B_{n-2}, \quad B_0 = 0, B_1 = 1, \tag{1.2}$$

$$H_n = 6H_{n-1} - H_{n-2}, \quad H_0 = 2, H_1 = 6, \tag{1.3}$$

$$C_n = 6C_{n-1} - C_{n-2}, \quad C_0 = 1, C_1 = 3. \tag{1.4}$$

The sequences $\{B_n\}_{n\geq 0}$,$\{H_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} B_{-n} &= 6B_{-(n-1)} - B_{-(n-2)}, \\ H_{-n} &= 6H_{-(n-1)} - H_{-(n-2)}, \\ C_{-n} &= 6C_{-(n-1)} - C_{-(n-2)}. \end{aligned}$$

for $n = 1, 2, 3, \cdots$ respectively.

Therefore recurrence equation (1.2), equation (1.3) and equation (1.4) hold for all integer n.

Next, we present the first few values of the balancing, modified Lucas-balancing and Lucas-balancing numbers with positive and negative subscripts:

Table 2. The first few values of the special second-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10
B_n	0	1	6	35	204	1189	6930	40391	235416	1372105	7997214
B_{-n}		-1	-6	-35	-204	-1189	-6930	-40391	-235416	-1372105	-7997214
H_n	2	6	34	198	1154	6726	39202	228486	1331714	7761798	45239074
H_{-n}		6	34	198	1154	6726	39202	228486	1331714	7761798	45239074
C_n	1	3	17	99	577	3363	19601	114243	665857	3880899	22619537
C_{-n}		3	17	99	577	3363	19 601	114243	665857	3880899	22619537

Binet's formula of generalized balancing numbers can be written as

$$W_n = \frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^n - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^n$$

where α and β are the roots of the quadratic equation

$$x^2 - 6x + 1 = 0.$$

Moreover

$$\begin{aligned} \alpha &= 3 + 2\sqrt[2]{2} \\ \beta &= 3 - 2\sqrt[2]{2}. \end{aligned}$$

Binet's formulas of balancing, modified Lucas-balancing and Lucas-balancing numbers are

$$\begin{aligned} B_n &= \frac{\alpha^n}{(\alpha-\beta)} + \frac{\beta^n}{(\alpha-\beta)}, \\ H_n &= \alpha^n + \beta^n, \\ C_n &= \frac{\alpha^n + \beta^n}{2}. \end{aligned}$$

respectively.

2. Preliminaries

A matrix $T = [t_{ij}] \in M_n(\mathbb{C})$ is called a Toeplitz matrix if it is of the form $t_{ij} = t_{i-j}$ for

$$T_n = \begin{pmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{1-n} \\ t_1 & t_0 & t_{-1} & \cdots & t_{2-n} \\ t_2 & t_1 & t_0 & \cdots & t_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_0 \end{pmatrix}.$$

Now, we give some preliminaries related to our study. Let $A = (a_{ij})$ be an $m \times n$ matrix. The ℓ_p norm of the matrix A is defined by

$$\|A\|_p = (\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^p)^{\frac{1}{p}} \qquad (1 \le p < \infty).$$

If $p = \infty$, then $\|A\|_\infty = \lim_{p\to\infty} \|A\|_p = \max_{i,j} |a_{ij}|$.

The well-known Frobenius (Euclidean) and spectral norms of the matrix A are defined respectively by

$$\|A\|_F = (\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2)^{\frac{1}{2}}$$

and

$$\|A\|_2 = \sqrt{\max_{1\le i\le n} |\lambda_i|} \tag{2.1}$$

where the numbers λ_i are the eigenvalues of matrix $A^H A$ and the matrix A^H is the conjugate transpose of the matrix A. The following inequality between the Frobenius and spectral norms of A holds.

$$\frac{1}{\sqrt{n}} \|A\|_F \le \|A\|_2 \le \|A\|_F. \tag{2.2}$$

It follows that

$$\|A\|_2 \le \|A\|_F \le \sqrt{n}\, \|A\|_2.$$

In literature, there are other types of norms of matrices. The maximum column sum matrix norm of $n \times n$ matrix $A = (a_{ij})$ is

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}| \tag{2.3}$$

and the maximum row sum matrix norm is

$$\|A\|_{\infty} = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}| \,. \tag{2.4}$$

The maximum column lenght norm $c_1(.)$ and maximum row lenght norm $r_1(.)$ of on matrix of order $m \times n$ are defined as follows

$$c_1(A) \equiv \max_{1\leq j\leq n} \left(\sum_{i=1}^{m} |a_{ij}|^2 \right)^{\frac{1}{2}} = \max_{1\leq j\leq n} \left\| [a_{ij}]_{i=1}^{m} \right\|_F \tag{2.5}$$

and

$$r_1(A) \equiv \max_{1\leq i\leq m} \left(\sum_{j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}} = \max_{1\leq i\leq m} \left\| [a_{ij}]_{j=1}^{n} \right\|_F \tag{2.6}$$

respectively.

For any $A, B \in M_{mn}(\mathbb{C})$, the Hadamard product of $A = (a_{ij})$ and $B = (b_{ij})$ is entrywise product and defined by $A \circ B = (a_{ij}b_{ij})$ and have the following properties

$$\|A \circ B\|_2 \leq r_1(A)\, c_1(B)\,, \tag{2.7}$$

and

$$\|A \circ B\|_2 \leq \|A\|_2 \|B\|_2 \,. \tag{2.8}$$

In addition,

$$\|A \circ B\|_F \leq \|A\|_F \|B\|_F \,. \tag{2.9}$$

Let $A \in M_{mn}(\mathbb{C})$, and $B \in M_{mn}(\mathbb{C})$ be given, then the Kronecker product of A, B is defined by

$$\|A \otimes B\| = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}$$

and have the following properties

$$\begin{aligned} \|A \otimes B\|_2 &= \|A\|_2 \|B\|_2 \,, \\ \|A \otimes B\|_F &= \|A\|_F \|B\|_F \,. \end{aligned} \tag{2.10}$$

In the following theorem, we present some sum formulas of generalized balancing numbers.

Theorem 1. *For generalized balancing numbers, we have following sum formulas:*

(a): *[6, Proposition 6.2. (a)] If* $x^2 - 6x + 1 \neq 0, i.e., x \neq 3 + 2\sqrt{2}, x \neq 3 - 2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_k = \frac{(x-6)x^{n+1}W_n + x^{n+1}W_{n-1} + (W_1 - 6W_0)x + W_0}{x^2 - 6x + 1}. \tag{2.11}$$

(b): *[6, Proposition 6.2 (d)] If* $x^2-6x+1\neq 0, i.e., x\neq 3+2\sqrt[2]{2}, x\neq 3-2\sqrt[2]{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-k} = \frac{x^{n+1}W_{-n+1}+(x-6)x^{n+1}W_{-n}-W_1x+W_0}{x^2-6x+1}.$$

(c): *[7, Proposition 2.2.(a)] If* $(x-1)(x^2-34x+1)\neq 0, i.e., x=1$ *or* $x=17-12\sqrt[2]{2}$ *or* $x=17+12\sqrt[2]{2}$, *then*

$$\sum_{k=0}^{n} x^k W_k^2 = \frac{\Psi}{3x^2-70x+35},$$

where

$\Psi=((x-34)x^{n+1}+(x-1)((n+2)x-34(n+1))x^nW_n^2+((n+2)x-(n+1))x^nW_{n-1}^2+W_0^2-(2x-1)(W_1-6W_0)^2+2(W_1^2+W_0^2-6W_1W_0)(x^n(n+1)-1)$.

(d): *[7, Proposition 2.2. .(d)] If* $(x-1)(x^2-34x+1)\neq 0, i.e., x=1$ *or* $x=17-12\sqrt{2}$ *or* $x=17+12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-k}^2 = \frac{\Psi}{(3x^2-70x+35)},$$

where

$\Psi=((n+2)x-(n+1))x^nW_{-n+1}^2+((x-34)x^{n+1}+(x-1)((n+2)x-34(n+1))x^n)W_{-n}^2+W_0^2-(2x-1)W_1^2+2(W_1^2+W_0^2-6W_1W_0)(x^n(n+1)-1)$.

If we set $x=1$ in the last Theorem, we obtained the following corollary.

Corollary 2. **??** *For generalized balancing numbers, we have following sum formulas:*

(a):

$$\sum_{k=0}^{n} W_k = \frac{5W_n-W_{n-1}+5W_0-W_1}{4}. \tag{2.12}$$

(b):

$$\sum_{k=0}^{n} W_{-k} = \frac{-W_{-n+1}+5W_{-n}+W_1-W_0}{4} \tag{2.13}$$

(c):

$$\sum_{k=0}^{n} W_k^2 = \frac{1}{32}(33W_n^2-W_{n-1}^2-W_0^2+(W_1-6W_0)^2-2n(W_1^2+W_0^2-6W_1W_0)). \tag{2.14}$$

(d):

$$\sum_{k=0}^{n} W_{-k}^2 = -\frac{1}{32}(W_{-n+1}^2-33W_{-n}^2+W_0^2-W_1^2+2n(W_1^2+W_0^2-6W_1W_0)). \tag{2.15}$$

3. Main Results

In this paper, we use the notation $A = T(W_0, W_1, \cdots, W_{n-1})$ for the Toeplitz matrix with generalized balancing numbers, i.e.,

$$A = \begin{pmatrix} W_0 & W_{-1} & W_{-2} & \cdots & W_{1-n} \\ W_1 & W_0 & W_{-1} & \cdots & W_{2-n} \\ W_2 & W_1 & W_0 & \cdots & W_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ W_{n-1} & W_{n-2} & W_{n-3} & \cdots & W_0 \end{pmatrix}. \tag{3.1}$$

For special cases, we get

$$A = \begin{pmatrix} B_0 & B_{-1} & B_{-2} & \cdots & B_{1-n} \\ B_1 & B_0 & B_{-1} & \cdots & B_{2-n} \\ B_2 & B_1 & B_0 & \cdots & B_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{n-1} & B_{n-2} & B_{n-3} & \cdots & B_0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -6 & \cdots & B_{1-n} \\ 1 & 0 & -1 & \cdots & B_{2-n} \\ 6 & 1 & 0 & \cdots & B_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{n-1} & B_{n-2} & B_{n-3} & \cdots & 0 \end{pmatrix} \tag{3.2}$$

for the Toeplitz matrix $A = T(B_0, B_1, \cdots, B_{n-1})$ with balancing numbers and

$$A = \begin{pmatrix} H_0 & H_{-1} & H_{-2} & \cdots & H_{1-n} \\ H_1 & H_0 & H_{-1} & \cdots & H_{2-n} \\ H_2 & H_1 & H_0 & \cdots & H_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{n-1} & H_{n-2} & H_{n-3} & \cdots & H_0 \end{pmatrix} = \begin{pmatrix} 2 & 6 & 34 & \cdots & H_{1-n} \\ 6 & 2 & 6 & \cdots & H_{2-n} \\ 34 & 6 & 2 & \cdots & H_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{n-1} & H_{n-2} & H_{n-3} & \cdots & 2 \end{pmatrix} \tag{3.3}$$

for the Toeplitz matrix $A = T(H_0, H_1, \cdots, H_{n-1})$ with modified Lucas-balancing numbers and

$$A = \begin{pmatrix} C_0 & C_{-1} & C_{-2} & \cdots & C_{1-n} \\ C_1 & C_0 & C_{-1} & \cdots & C_{2-n} \\ C_2 & C_1 & C_0 & \cdots & C_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{n-1} & C_{n-2} & C_{n-3} & \cdots & C_0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 17 & \cdots & C_{1-n} \\ 3 & 1 & 3 & \cdots & C_{2-n} \\ 17 & 3 & 1 & \cdots & C_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{n-1} & C_{n-2} & C_{n-3} & \cdots & 1 \end{pmatrix} \tag{3.4}$$

for the Toeplitz matrix $A = T(C_0, C_1, \cdots, C_{n-1})$ with Lucas-balancing numbers.

In the following theorem, we present the norm value of $\|A\|_1$ and $\|A\|_\infty$ of the largest absolute column sum and the largest absolute row sum of A.

Theorem 3. *Let $A = T(W_0, W_1, \cdots, W_{n-1})$ be a Toeplitz matrix with generalized balancing numbers then the largest absolute column sum (1-norm) and the largest absolute row sum (∞-norm) of A are*

$$\|A_1\| = \|A\|_\infty = \begin{cases} \frac{1}{4}(-W_n + W_{n-1} - 5W_0 + W_1) & , \quad \text{if } |W_k| = |W_{-k}| \quad \text{and } W_k \leq 0 \\ \frac{1}{4}(W_n - W_{n-1} + 5W_0 - W_1) & , \quad \text{if } |W_k| = |W_{-k}| \quad \text{and } W_k \geq 0 \end{cases}$$

where $k = i - j : i, j = 0, 1, \cdots, n-1$; $(k \in N, -k \in N^{-})$.

Proof. Consider $A = T(W_0, W_1, \cdots, W_{n-1})$ which is given as in (3.1). By the definitions of $1 - norm$ and $\infty - norm$, and equation (2.3) and equation (2.4) and equation (2.12), we conclude that

(i): If $|W_k| = |W_{-k}|$, $k \in N$ and $W_k \leq 0$, $k \in N$, then we get

$$\begin{aligned} \|A\|_1 &= \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}| = \max\{|a_{1j}| + |a_{2j}| + |a_{3j}| + \cdots + |a_{nj}|\} \\ &= |a_{11}| + |a_{21}| + |a_{31}| + \cdots + |a_{n1}| \quad = \sum_{i=1}^{n} |a_{i1}| \\ &= \sum_{k=0}^{n-1} |W_k| = -\sum_{k=0}^{n-1} W_k = -\sum_{k=0}^{n} W_k + W_n \\ &= \frac{-W_n + W_{n-1} - 5W_0 + W_1}{4} \end{aligned}$$

and if $|W_k| = |W_{-k}|$, $k \in N$ and $W_k \geq 0$, $k \in N$, then we obtain

$$\begin{aligned} \|A\|_1 &= \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}| = \max\{|a_{1j}| + |a_{2j}| + |a_{3j}| + \cdots + |a_{nj}|\} \\ &= |a_{11}| + |a_{21}| + |a_{31}| + \cdots + |a_{n1}| \quad = \sum_{i=1}^{n} |a_{i1}| \\ &= \sum_{k=0}^{n-1} |W_k| = \sum_{k=0}^{n-1} W_k = \sum_{k=0}^{n} W_k - W_n \\ &= \frac{W_n - W_{n-1} + 5W_0 - W_1}{4}. \end{aligned}$$

(ii): If $|W_k| = |W_{-k}|$, $k \in N$ and $W_k \leq 0$, $k \in N$, then we get

$$\begin{aligned} \|A\|_\infty &= \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}| = \max\{|a_{i1}| + |a_{i2}| + |a_{i3}| + \cdots + |a_{in}|\} \\ &= |a_{n1}| + |a_{n2}| + \cdots + |a_{nn}| = \sum_{j=1}^{n} |a_{nj}| \\ &= \sum_{k=0}^{n-1} |W_k| = -\sum_{k=0}^{n-1} W_k = -\sum_{k=0}^{n} W_k + W_n \\ &= \frac{-W_n + W_{n-1} - 5W_0 + W_1}{4} \end{aligned}$$

and if $|W_k| = |W_{-k}|$, $k \in N$ and $W_k \geq 0$, $k \in N$, then we obtain

$$\begin{aligned}
\|A\|_\infty &= \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}| = \max\{|a_{i1}| + |a_{i2}| + |a_{i3}| + \cdots + |a_{in}|\} \\
&= |a_{n1}| + |a_{n2}| + \cdots + |a_{nn}| = \sum_{j=1}^{n} |a_{nj}| \\
&= \sum_{k=0}^{n-1} |W_k| = \sum_{k=0}^{n-1} W_k = \sum_{k=0}^{n} W_k - W_n \\
&= \frac{W_n - W_{n-1} + 5W_0 - W_1}{4}.
\end{aligned}$$

Thus, the proof is completed.□

Remark 4. *In the statement of the theorem 3 the condion on W_n, W_{-n}, $n \in N$ is given to calculade $\|.\|_1$ and $\|.\|_\infty$ norms of balancing, modified Lucas-balancing, Lucas-balancing numbers. The other cases can be handled similarly.*

From the last Theorem 3, we have the following corollary which present norm values of $\|A\|_1, \|A\|_\infty$ of the largest absolute column sum and the largest absolute row sum of A with balancing numbers, modified Lucas-balancing numbers and Lucas-balancing numbers, respectively, (set $W_n = B_n$ with $B_0 = 0, B_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 6$ and $W_n = C_n$ with $C_0 = 1, C_1 = 3$, respectively).

Corollary 5.

- **(a):** For $A = T(B_0, B_1, \cdots, B_{n-1})$, the vaules of norms of Toeplitz matrices with balancing numbers have the following property:

$$\|A\|_1 = \|A\|_\infty = \frac{B_n - B_{n-1} - 1}{4}.$$

- **(b):** For $A = T(H_0, H_1, \cdots, H_{n-1})$, the values of norms of Toeplitz matrices with modified Lucas-balancing numbers have the following property:

$$\|A\|_1 = \|A\|_\infty = \frac{H_n - H_{n-1} + 4}{4}.$$

- **(c):** For $A = T(C_0, C_1, \cdots, C_{n-1})$, the values of norms of Toeplitz matrices with Lucas-balancing numbers have the following property:

$$\|A\|_1 = \|A\|_\infty = \frac{C_n - C_{n-1} + 2}{4}.$$

Next theorem presents the Frobenious (Euclidian) norm of a Toeplitz matrix A.

Theorem 6. *Let $A = T(W_0, W_1, \cdots, W_{n-1})$ be a Toeplitz matrix, then the Frobenious (Euclidian) norm of matrix A is*

$$\|A\|_F = \sqrt{\Upsilon_1}$$

where $\Upsilon_1 = \frac{1}{32}W_n^2 + \frac{1}{32}W_{-n}^2 - \frac{(34n+2)}{32}W_0^2 + \frac{n}{32}W_1^2 + \frac{n}{32}(W_1 - 6W_0)^2 - \frac{2n(n+1)}{32}(W_1^2 + W_0^2 - 6W_1W_0)$.

Proof. The matrix A is of the form

$$A = \begin{pmatrix} W_0 & W_{-1} & W_{-2} & \cdots & W_{1-n} \\ W_1 & W_0 & W_{-1} & \cdots & W_{2-n} \\ W_2 & W_1 & W_0 & \cdots & W_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ W_{n-1} & W_{n-2} & W_{n-3} & \cdots & W_0 \end{pmatrix}.$$

Then we have

$$\begin{aligned} \|A\|_F^2 &= nW_0^2 + (n-1)W_{-1}^2 + (n-2)W_{-2}^2 + (n-3)W_{-3}^2 + \cdots + W_{1-n}^2 \\ &\quad +(n-1)W_1^2 + (n-2)W_2^2 + (n-3)W_3^2 + \cdots + W_{n-1}^2 \end{aligned}$$

we obtain and it follows that

$$\begin{aligned} \|A\|_F^2 &= nW_0^2 + (n-1)W_{-1}^2 + (n-2)W_{-2}^2 + (n-3)W_{-3}^2 + \cdots + W_{1-n}^2 \\ &\quad +(n-1)W_1^2 + (n-2)W_2^2 + (n-3)W_3^2 + \cdots + W_{n-1}^2 \\ &= nW_0^2 + \sum_{k=1}^{n-1}\sum_{i=1}^{k} W_i^2 + \sum_{k=1}^{n-1}\sum_{i=1}^{k} W_i^2 \\ &= (2-n)W_0^2 + \sum_{k=1}^{n-1}(\frac{1}{32}(33W_n^2 - W_{n-1}^2 - W_0^2 + (W_1 - 6W_0)^2 - 2n(W_1^2 + W_0^2 - 6W_1W_0))) \\ &\quad + \sum_{k=1}^{n-1}(-\frac{1}{32}(W_{-n+1}^2 - 33W_{-n}^2 + W_0^2 - W_1^2 + 2n(W_1^2 + W_0^2 - 6W_1W_0))) \\ &= (2-n)W_0^2 + \frac{1}{32}(W_n^2 - (n+33)W_0^2 + n(W_1 - 6W_0)^2 - n(n+1)(W_1^2 + W_0^2 - 6W_1W_0)) \\ &\quad + \frac{1}{32}W_{-n}^2 - \frac{(n+33)}{32}W_0^2 + \frac{n}{32}W_1^2 - \frac{n(n+1)}{32}(W_1^2 + W_0^2 - 6W_1W_0) \\ &= \frac{1}{32}W_n^2 + \frac{1}{32}W_{-n}^2 - \frac{(34n+2)}{32}W_0^2 + \frac{n}{32}W_1^2 + \frac{n}{32}(W_1 - 6W_0)^2 - \frac{2n(n+1)}{32}(W_1^2 + W_0^2 - 6W_1W_0). \end{aligned}$$

Moreover, we use equation 2.14 and equation 2.15 in Corollary **??**.

Therefore, we get

$$\|A\|_F^2 = \frac{1}{32}W_n^2 + \frac{1}{32}W_{-n}^2 - \frac{(34n+2)}{32}W_0^2 + \frac{n}{32}W_1^2 + \frac{n}{32}(W_1 - 6W_0)^2 - \frac{2n(n+1)}{32}(W_1^2 + W_0^2 - 6W_1W_0).$$

This complates the proof. □

From the last Theorem 6, we have the following corollary which gives Frobenius norm formulas of balancing numbers, modified Lucas-balancing numbers and Lucas-balancing numbers, respectively, (take $W_n = B_n$ with $B_0 = 0, B_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 6$ and $W_n = C_n$ with $C_0 = 1, C_1 = 3$, respectively).

Corollary 7. *For $n \geq 0$, Toeplitz matrices with the balancing, modified Lucas-balancing and Lucas-balancing numbers, recpectively, have the following properties:*

(a): $\|A\|_F = \sqrt{\Upsilon_2}$
where A is given as in (3.2)

$$\Upsilon_2 = \frac{1}{32}(B_n^2 + B_{-n}^2 - 2n^2).$$

(b): $\|A\|_F = \sqrt{\Upsilon_3}$
where A is given as in (3.3)

$$\Upsilon_3 = \frac{1}{32}(H_n^2 + H_{-n}^2 + (64n^2 - 8)).$$

(c): $\|A\|_F = \sqrt{\Upsilon_4}$
where A is given as in (3.4)

$$\Upsilon_4 = \frac{1}{32}(C_n^2 + C_{-n}^2 + 16n^2 - 2).$$

In the following theorem, we find the lower and upper bounds for the spectral norms of the matrices with the balancing numbers, modified Lucas-balancing numbers and Lucas-balancing numbers, respectively, (take $W_n = B_n$ with $B_0 = 0, B_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 6$ and $W_n = C_n$ with $C_0 = 1, C_1 = 3$, respectively).

Theorem 8.

(a): Consider $A = T(B_0, B_1, \cdots, B_{n-1})$ which is given as in (3.2). Let

$$C = \begin{pmatrix} 1 & B_{-1} & B_{-2} & \cdots & B_{1-n} \\ 1 & B_0 & B_{-1} & \cdots & B_{2-n} \\ 1 & B_1 & B_0 & \cdots & B_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & B_{n-2} & B_{n-3} & \cdots & B_0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -6 & \cdots & B_{1-n} \\ 1 & 0 & -1 & \cdots & B_{2-n} \\ 1 & 1 & 0 & \cdots & B_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & B_{n-2} & B_{n-3} & \cdots & 0 \end{pmatrix}$$

and

$$D = \begin{pmatrix} B_0 & 1 & 1 & \cdots & 1 \\ B_1 & 1 & 1 & \cdots & 1 \\ B_2 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 6 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix}$$

such that $A = C \circ D$ (Hadamart Product of C and D).

(i):

$$\|A\|_2 \geq \sqrt{\frac{1}{n}\Upsilon_2}$$

where Υ_2 is as in Corollary 7.

(ii):

$$\|A\|_2 \leq \Upsilon_5$$

where

$$\Upsilon_5 = (\frac{1}{32}(B_n^2 - 33B_{n-1}^2 - 2n + 33))^{\frac{1}{2}} \times (\frac{1}{32}(B_n^2 - B_{n-1}^2 + 1 - 2n))^{\frac{1}{2}}.$$

(b): Consider $A = T(H_0, H_1, \cdots, H_{n-1})$ which is given as in (3.3). Let

$$C = \begin{pmatrix} 1 & H_{-1} & H_{-2} & \cdots & H_{1-n} \\ 1 & H_0 & H_{-1} & \cdots & H_{2-n} \\ 1 & H_1 & H_0 & \cdots & H_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & H_{n-2} & H_{n-3} & \cdots & H_0 \end{pmatrix} = \begin{pmatrix} 1 & 6 & 34 & \cdots & H_{1-n} \\ 1 & 2 & 6 & \cdots & H_{2-n} \\ 1 & 6 & 2 & \cdots & H_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & H_{n-2} & H_{n-3} & \cdots & 2 \end{pmatrix}$$

and

$$D = \begin{pmatrix} H_0 & 1 & 1 & \cdots & 1 \\ H_1 & 1 & 1 & \cdots & 1 \\ H_2 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 6 & 1 & 1 & \cdots & 1 \\ 34 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix}$$

such that $A = C \circ D$ (Hadamart Product of C and D).

(i):

$$\|A\|_2 \geq \sqrt{\frac{1}{n}\Upsilon_3}$$

where Υ_3 is as in Corollary 7.

(ii):

$$\|A\|_2 \leq \Upsilon_6$$

where

$$\Upsilon_6 = (2 + \frac{1}{32}(H_n^2 - 33H_{n-1}^2 + 64n))^{\frac{1}{2}} \times (\frac{1}{32}(H_n^2 - H_{n-1}^2 + 32 + 64n))^{\frac{1}{2}}.$$

(c): Consider $A = T(C_0, C_1, \cdots, C_{n-1})$ which is given as in (3.4). Let

$$C = \begin{pmatrix} 1 & C_{-1} & C_{-2} & \cdots & C_{1-n} \\ 1 & C_0 & C_{-1} & \cdots & C_{2-n} \\ 1 & C_1 & C_0 & \cdots & C_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & C_{n-2} & C_{n-3} & \cdots & C_0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 17 & \cdots & C_{1-n} \\ 1 & 1 & 3 & \cdots & C_{2-n} \\ 1 & 3 & 1 & \cdots & C_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & C_{n-2} & C_{n-3} & \cdots & 1 \end{pmatrix}$$

and

$$D = \begin{pmatrix} C_0 & 1 & 1 & \cdots & 1 \\ C_1 & 1 & 1 & \cdots & 1 \\ C_2 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 3 & 1 & 1 & \cdots & 1 \\ 17 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix}$$

such that $A = C \circ D$ (Hadamart Product of C and D).

(i):

$$\|A\|_2 \geq \sqrt{\frac{1}{n}\Upsilon_4}$$

where Υ_4 is as in Corollary 7.

(ii):

$$\|A\|_2 \leq \Upsilon_7$$

where

$$\Upsilon_7 = (\frac{1}{32}(C_n^2 - 33C_{n-1}^2 + 40 + 16n))^{\frac{1}{2}} \times (\frac{1}{32}(C_n^2 - C_{n-1}^2 + 8 + 16n))^{\frac{1}{2}}.$$

Proof.

(a): **(i):** We use equation (2.2).

(ii): We get

$$\begin{aligned} r_1(C) &= \max_i (\sum_j |c_{ij}|^2)^{\frac{1}{2}} = (\sum_{j=1}^{n} |c_{nj}|^2)^{\frac{1}{2}} = (1 + \sum_{k=0}^{n-2} B_k^2)^{\frac{1}{2}} \\ &= (1 + \frac{1}{32}(B_n^2 - 33B_{n-1}^2 - B_0^2 + (B_1 - 6B_0)^2 \\ &\quad -2n(B_1^2 + B_0^2 - 6B_1B_0)))^{\frac{1}{2}} \\ &= (\frac{1}{32}(B_n^2 - 33B_{n-1}^2 - 2n + 33))^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} c_1(D) &= \max_j (\sum_i |d_{ij}|^2)^{\frac{1}{2}} = (\sum_{i=1}^{n} |d_{i1}|^2)^{\frac{1}{2}} \\ &= (\sum_{k=0}^{n-1} B_k^2)^{\frac{1}{2}} = (\sum_{k=0}^{n} B_k^2 - B_n^2)^{\frac{1}{2}} \\ &= ((\frac{1}{32}(33B_n^2 - B_{n-1}^2 - B_0^2 + (B_1 - 6B_0)^2 \\ &\quad -2n(B_1^2 + B_0^2 - 6B_1B_0)) - B_n^2)^{\frac{1}{2}} \\ &= (\frac{1}{32}(B_n^2 - B_{n-1}^2 - B_0^2 + (B_1 - 6B_0)^2 \\ &\quad -2n(B_1^2 + B_0^2 - 6B_1B_0)))^{\frac{1}{2}} \\ &= (\frac{1}{32}(B_n^2 - B_{n-1}^2 + 1 - 2n))^{\frac{1}{2}} \end{aligned}$$

so, from inequality (2.7),

$$\begin{aligned}\|A\|_2 &\leq r_1(C)c_1(D) = \Upsilon_5\\ &= (\frac{1}{32}(B_n^2 - 33B_{n-1}^2 - 2n + 33))^{\frac{1}{2}}\\ &\quad\times(\frac{1}{32}(B_n^2 - B_{n-1}^2 + 1 - 2n))^{\frac{1}{2}}.\end{aligned}$$

(b): **(i):** We use equation (2.2).

(ii): By definition, we get

$$\begin{aligned}r_1(C) &= \max_i(\sum_j |c_{ij}|^2)^{\frac{1}{2}} = (\sum_{j=1}^{n} |c_{nj}|^2)^{\frac{1}{2}} = (1 + \sum_{k=0}^{n-2} H_k^2)^{\frac{1}{2}}\\ &= (1 + \frac{1}{32}(H_n^2 - 33H_{n-1}^2 - H_0^2 + (H_1 - 6H_0)^2\\ &\quad -2n(H_1^2 + H_0^2 - 6H_1H_0)))^{\frac{1}{2}}\\ &= (2 + \frac{1}{32}(H_n^2 - 33H_{n-1}^2 + 64n))^{\frac{1}{2}}\end{aligned}$$

and

$$\begin{aligned}c_1(D) &= \max_j(\sum_i |d_{ij}|^2)^{\frac{1}{2}} = (\sum_{i=1}^{n} |d_{i1}|^2)^{\frac{1}{2}}\\ &= (\sum_{k=0}^{n-1} H_k^2)^{\frac{1}{2}} = (\sum_{k=0}^{n} H_k^2 - H_n^2)^{\frac{1}{2}}\\ &= ((\frac{1}{32}(33H_n^2 - H_{n-1}^2 - H_0^2 + (H_1 - 6H_0)^2\\ &\quad -2n(H_1^2 + H_0^2 - 6H_1H_0)) - H_n^2)^{\frac{1}{2}}\\ &= (\frac{1}{32}(H_n^2 - H_{n-1}^2 - 4 + 36 - 2n(36 + 4 - 72)))^{\frac{1}{2}}\\ &= (\frac{1}{32}(H_n^2 - H_{n-1}^2 + 32 + 64n))^{\frac{1}{2}}.\end{aligned}$$

so, from inequality (2.7),

$$\begin{aligned}\|A\|_2 &\leq r_1(C)c_1(D) = \Upsilon_6\\ &= (2 + \frac{1}{32}(H_n^2 - 33H_{n-1}^2 + 64n))^{\frac{1}{2}}\\ &\quad\times(\frac{1}{32}(H_n^2 - H_{n-1}^2 + 32 + 64n))^{\frac{1}{2}}.\end{aligned}$$

(c): **(i):** We use equation (2.2).

(ii): By definition, we get

$$\begin{aligned} r_1(C) &= \max_i (\sum_j |c_{ij}|^2)^{\frac{1}{2}} = (\sum_{j=1}^{n} |c_{nj}|^2)^{\frac{1}{2}} = (1 + \sum_{k=0}^{n-2} C_k^2)^{\frac{1}{2}} \\ &= (1 + \frac{1}{32}(C_n^2 - 33C_{n-1}^2 - C_0^2 + (C_1 - 6C_0)^2 \\ &\quad -2n(C_1^2 + C_0^2 - 6C_1C_0)))^{\frac{1}{2}} \\ &= (\frac{1}{32}(C_n^2 - 33C_{n-1}^2 + 40 + 16n))^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} c_1(D) &= \max_j (\sum_i |d_{ij}|^2)^{\frac{1}{2}} = (\sum_{i=1}^{n} |d_{i1}|^2)^{\frac{1}{2}} \\ &= (\sum_{k=0}^{n-1} C_k^2)^{\frac{1}{2}} = (\sum_{k=0}^{n} C_k^2 - C_n^2)^{\frac{1}{2}} \\ &= ((\frac{1}{32}(33C_n^2 - C_{n-1}^2 - C_0^2 + (C_1 - 6C_0)^2 \\ &\quad -2n(C_1^2 + C_0^2 - 6C_1C_0)) - C_n^2)^{\frac{1}{2}} \\ &= (\frac{1}{32}(C_n^2 - C_{n-1}^2 - 1 + (3-6)^2 - 2n(9 + 1 - 18)))^{\frac{1}{2}} \\ &= (\frac{1}{32}(C_n^2 - C_{n-1}^2 + 8 + 16n))^{\frac{1}{2}}. \end{aligned}$$

so, by definition of Hadamard product and from inequality (2.7),

$$\begin{aligned} \|A\|_2 &\leq r_1(C)c_1(D) = \Upsilon_7 \\ &= (\frac{1}{32}(C_n^2 - 33C_{n-1}^2 + 40 + 16n))^{\frac{1}{2}} \\ &\quad \times(\frac{1}{32}(C_n^2 - C_{n-1}^2 + 8 + 16n))^{\frac{1}{2}}. \end{aligned}$$

This complates the proof.□

From the equation (2.10) and Corollary 7, we have the following corollary which gives the Frobenius norms of the Kronecker products of the Toeplitz matrices with special cases of generalized balancing numbers.

Corollary 9.

(a): Let $A = T(B_0, B_1, \cdots, B_{n-1})$ and $B = T(H_0, H_1, \cdots, H_{n-1})$ be Toeplitz matrices with balancing numbers and modified Lucas-balancing numbers, respectively, then we have the following property:

$$\begin{aligned} \|A \otimes B\|_F &= \|A\|_F \|B\|_F \\ &= \sqrt{\Upsilon_2}\sqrt{\Upsilon_3} \end{aligned}$$

where Υ_2 and Υ_3 are as in Corollary 7, (a) and (b),

(set $W_n = B_n$ with $B_0 = 0, B_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 6$, respectively).

(b): Suppose that $A = T(B_0, B_1, \cdots, B_{n-1})$ and $B = T(C_0, C_1, \cdots, C_{n-1})$ be Toeplitz matrices with balancing numbers and modified Lucas-balancing numbers, respectively, then we obtain the following property:

$$\begin{aligned} \|A \otimes B\|_F &= \|A\|_F \|B\|_F \\ &= \sqrt{\Upsilon_2}\sqrt{\Upsilon_4} \end{aligned}$$

where Υ_2 and Υ_4 are as in Corollary 7, (a) and (c),
(set $W_n = B_n$ with $B_0 = 0, B_1 = 1$ and $W_n = C_n$ with $C_0 = 1, C_1 = 3$, respectively).

(c): Given $A = T(H_0, H_1, \cdots, H_{n-1})$ and $B = T(C_0, C_1, \cdots, C_{n-1})$ be Toeplitz matrices with modified Lucas-balancing numbers and Lucas-balancing numbers respectively, then we get the following property:

$$\begin{aligned} \|A \otimes B\|_F &= \|A\|_F \|B\|_F \\ &= \sqrt{\Upsilon_3}\sqrt{\Upsilon_4} \end{aligned}$$

where Υ_3 and Υ_4 are as in Corollary 7, (b) and (c),
(set $W_n = H_n$ with $H_0 = 2, H_1 = 6$ and $W_n = C_n$ with $C_0 = 1, C_1 = 3$, respectively).

Proof. (a), (b) and (c) follows from equation (2.10) and Theorem 6 and Corollary 7.

From the above inequality (2.9) and Theorem 6 and Corollary 7, we have the following result, which gives an upper bound for the Frobenius norm of Hadamard products of Toeplitz matrices by exclusive cases of generalized balancing numbers.

Corollary 10.

(a): Let $A = T(B_0, B_1, \cdots, B_{n-1})$ and $B = T(H_0, H_1, \cdots, H_{n-1})$ be Toeplitz matrices with balancing numbers and modified Lucas-balancing numbers, respectively, then we have the following property:

$$\begin{aligned} \|A \circ B\|_F &\leq \|A\|_F \|B\|_F \\ &\leq \sqrt{\Upsilon_2}\sqrt{\Upsilon_3} \end{aligned}$$

where Υ_2 and Υ_3 are as in Corollary 7, (a) and (b),
(set $W_n = B_n$ with $B_0 = 0, B_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 6$, respectively).

(b): Suppose that $A = T(B_0, B_1, \cdots, B_{n-1})$ and $B = T(C_0, C_1, \cdots, C_{n-1})$ be Toeplitz matrices with balancing numbers and Lucas-balancing numbers, respectively, then we obtain the following property:

$$\begin{aligned} \|A \circ B\|_F &\leq \|A\|_F \|B\|_F \\ &\leq \sqrt{\Upsilon_2}\sqrt{\Upsilon_4} \end{aligned}$$

where Υ_2 and Υ_4 are as in Corollary 7, (a) and (c), (set $W_n = B_n$ with $B_0 = 0, B_1 = 1$ and $W_n = C_n$ with $C_0 = 1, C_1 = 3$, respectively).

(c): Assume that $A = T(H_0, H_1, \cdots, H_{n-1})$ and $B = T(C_0, C_1, \cdots, C_{n-1})$ be Toeplitz matrices with modified Lucas-balancing numbers and Lucas-balancing numbers, respectively, then we have the following property:

$$\begin{aligned} \|A \circ B\|_F &\leq \|A\|_F \|B\|_F \\ &\leq \sqrt{\Upsilon_3}\sqrt{\Upsilon_4} \end{aligned}$$

where Υ_3 and Υ_4 are as in Corollary 7, (a) and (c),

(set $W_n = H_n$ with $H_0 = 2, H_1 = 6$ and $W_n = C_n$ with $C_0 = 1, C_1 = 3$, respectively).

In the last inequality (2.8) and Theorem 8, we have the following corollary, which gives an upper bound for the spectral norm of Hadamard products of Toeplitz matrices with special cases of generalized balancing numbers.

Corollary 11.

(a): Given $A = T(B_0, B_1, \cdots, B_{n-1})$ and $B = T(H_0, H_1, \cdots, H_{n-1})$ be Toeplitz matrices with balancing numbers and modified Lucas-balancing numbers respectively, then we have the following property:

$$\|A \circ B\|_2 \leq \Upsilon_5 \times \Upsilon_6$$

where Υ_5 and Υ_6 are as in Theorem 8,

(take $W_n = B_n$ with $B_0 = 0, B_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 6$, respectively).

(b): Let $A = T(B_0, B_1, \cdots, B_{n-1})$ and $B = T(C_0, C_1, \cdots, C_{n-1})$ be Toeplitz matrices with balancing numbers and Lucas-balancing numbers respectively, then we have the following property:

$$\|A \circ B\|_2 \leq \Upsilon_5 \times \Upsilon_7$$

where Υ_5 and Υ_7 are as in Theorem 8,

(set $W_n = B_n$ with $B_0 = 0, B_1 = 1$ and $W_n = C_n$ with $C_0 = 1, C_1 = 3$, respectively).

(c): Suppose that $A = T(H_0, H_1, \cdots, H_{n-1})$ and $B = T(C_0, C_1, \cdots, C_{n-1})$ be Toeplitz matrices with modified Lucas-balancing numbers and Lucas-balancing numbers, respectively, then we get the following property:

$$\|A \circ B\|_2 \leq \Upsilon_6 \times \Upsilon_7$$

where Υ_6 and Υ_7 are as in Theorem 8,

(set $W_n = H_n$ with $H_0 = 2, H_1 = 6$ and $W_n = C_n$ with $C_0 = 1, C_1 = 3$, respectively).

Proof. For (a), (b) and (c) see equation (2.8) and Theorem 8.

From the related equation (2.10) and Theorem 8, we have the following corollary which gives an upper bound for the spectral norm of Kronocker products of Toeplitz matrices with special cases of generalized balancing numbers.

(a): Let $A = T(B_0, B_1, \cdots, B_{n-1})$ and $B = T(H_0, H_1, \cdots, H_{n-1})$ be Toeplitz matrices with balancing numbers and modified Lucas-balancing numbers respectively, then we have the following property:

$$\|A \otimes B\|_2 \leq \Upsilon_5 \times \Upsilon_6$$

where Υ_5 and Υ_6 are as in Theorem 8 ,

(set $W_n = B_n$ with $B_0 = 0, B_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 6$, respectively).

(b): Let $A = T(B_0, B_1, \cdots, B_{n-1})$ and $B = T(C_0, C_1, \cdots, C_{n-1})$ be Toeplitz matrices with balancing numbers and Lucas-balancing numbers, respectively, then we get the following property:

$$\|A \otimes B\|_2 \leq \Upsilon_5 \times \Upsilon_7$$

where Υ_5 and Υ_7 are as in Theorem 8,

(set $W_n = B_n$ with $B_0 = 0, B_1 = 1$ and $W_n = C_n$ with $C_0 = 1, C_1 = 3$, respectively).

(c): Let $A = T(H_0, H_1, \cdots, H_{n-1})$ and $B = T(C_0, C_1, \cdots, C_{n-1})$ be Toeplitz matrices with modified Lucas-balancing numbers and Lucas-balancing numbers, respectively, then we obtain the following property:

$$\|A \otimes B\|_2 \leq \Upsilon_6 \times \Upsilon_7$$

where Υ_6 and Υ_7 are as in Theorem 8,

(set $W_n = H_n$ with $H_0 = 2, H_1 = 6$ and $W_n = C_n$ with $C_0 = 1, C_1 = 3$, respectively).

Proof. For (a), (b) and (c) see equation (2.10) and Theorem 8.

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